Math 261B Tues 10/13

Sp_{2n} vs. SO_{2n}

Sp_{2n} preserves $x^T J_- y$ $\quad J = \begin{pmatrix} & & & & 1 \\ & & & \iddots & \\ & & 1 & & \\ & -1 & & & \\ \iddots & & & & \\ -1 & & & & \end{pmatrix}$

$SO_N \subset SL_N$ preserves $x^T J y$ $\quad J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$

If char $K = 2$ $\quad (J_-)_{2n} = (J)_{2n}$ $\Rightarrow$ "SO_{2n}" as we defined it is Sp_{2n}.

(SO_{2n+1} as we defined it is non-reduced)
$A^R A = I$, $\det A = 1$

Fix If char $K \neq 2$

Quadratic forms $\longleftrightarrow$ symmetric bilinear forms (on K^N)

$Q(x_1, \ldots, x_N)$

$$Q(x) \longmapsto (x,y) = Q(x+y) - Q(x) - Q(y)$$

$$(\, , \,) \longmapsto Q(x) = (x,x)$$

$$Q \mapsto (\, , \,) \longmapsto 2Q \longmapsto 2(\, , \,)$$

$$(x,x) = Q(x+x) - Q(x) - Q(x) = 4Q(x) - 2Q(x) = 2Q(x)$$

If $\operatorname{char} k = 2$ have $Q \mapsto (\, , \,)$; $(\, , \,) \mapsto Q$ not inverse!

$$\uparrow$$

$$x^T J_{2n} y \mapsto Q(x) = x^T J_{2n} x$$

$$= 2(x_1 x_{2n} + \cdots + x_n x_{n+1})$$

$$= 0 \text{ if } \operatorname{char} k = 2.$$

Classification of quadratic forms over $k = \overline{k}$ up to linear change of variables

- Unique Q s.t. $V(Q) \subset \mathbb{P}^{N-1}$ is non-singular, or $V(Q) \subset \mathbb{A}^N$ is singular only at 0.

Standard form: $x_1 x_{2n} + \cdots + x_n x_{n+1}$ $\quad N = 2n$

$x_1 x_{2n+1} + x_2 x_{2n} + \cdots + x_{n-1} x_{n+2} + x_{n+2}^2$ $\quad N = 2n+1$

- others are the standard Q in proper subset of the variables. (they are singular where $x_i = 0$ for x_i not in Q)

Right def'n of O_N is group preserving a standard $Q(x)$

$$SO_N = O_N \cap SL_N$$

Standard $Q \to (x,y) = x^T \tilde{J} y$ $\qquad \tilde{J} = J$ for $N=2n$

$$\tilde{J} = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 2 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

$A = O_N$ if $Q(Ax) = Q(x)$

$\rightsquigarrow A^{-1}Q = Q$

$$PGl_2 \to SO_{2n+1}$$

Ex. O_2, SO_2

$Q(x) = x_1 x_2$ $\qquad K^3 = sl_2 = \langle E, H, F \rangle$

$Q\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = (ax_1 + bx_2) \cdot (cx_1 + dx_2)$ $\qquad -\det\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = bc + a^2$

$b = x_1 \quad a = x_2 \quad c = x_3$

$$= ac\,x_1^2 + (ad+bc)\,x_1 x_2 + bd\,x_2^2 = x_1 x_2$$

$\Rightarrow$ $ac = 0, \quad bd = 0, \quad ad + bc = 1$

If $a = 0$, $bc = 1 \Rightarrow d = 0$ $\qquad \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ $bc = 1$ $\quad \det = -1$

If $c = 0$, $ad = 1 \Rightarrow b = 0$ $\qquad \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ $ad = 1$ $\quad \det = 1$

$O_2 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \amalg \begin{pmatrix} 0 & b \\ b^{-1} & 0 \end{pmatrix}$ $\qquad SO_2 \quad \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cong \mathbb{G}_m$

det = 1 $\qquad$ det = −1

$ad + bc = 1 \Rightarrow \det A = 2ad - 1 \qquad ad = \frac{1+\det A}{2}$

ad is 1 on SO_2, 0 on the other coset $\qquad bc = \frac{1-\det A}{2}$

bc is 0 on SO_2, 1 on the other coset.

$bc = 0$ $\rightsquigarrow$ ideal these generate is $(b, c, ad-1)$

Ex. O_3 / SO_3 Eq'ns by equating $Q(Ax) = Q(x)$ = value of LHS at $A = I$.

k^3

$(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)\cdot(a_{31}x_1 + a_{32}x_2 + a_{33}x_3)$
$+ (a_{21}x_1 + a_{22}x_2 + a_{23}x_3)^2$

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$Q = x_1 x_3 + x_2^2$

$\langle x_1 x_2 \rangle$: $a_{11} a_{32} + a_{12} a_{31} + 2 a_{21} a_{22} = 0$

$\langle x_1 x_3 \rangle$: $a_{11} a_{33} + a_{13} a_{31} + 2 a_{21} a_{23} = 1$

$\langle x_2 x_3 \rangle$: $a_{12} a_{33} + a_{13} a_{32} + 2 a_{22} a_{23} = 0$

$\langle x_1^2 \rangle$: $a_{11} a_{31} + a_{21}^2 = 0$

$\langle x_2^2 \rangle$: $a_{12} a_{32} + a_{22}^2 = 1$

$\langle x_3^2 \rangle$: $a_{13} a_{33} + a_{23}^2 = 0$

eqn's of O_3

Add $\det A = 1$ for SO_3

$A^R A = I$ $(\det A = 1)$

$M^R + M = 0$ for so_3

$$\begin{pmatrix} & x & ? \\ & ? & -x \\ ? & & \end{pmatrix}$$

Calculate M correctly:

$A = I + \varepsilon M$ $\qquad \varepsilon^2 = 0$

$M_{32} + 2M_{21} = 0$

$M_{33} + M_{11} = 0$

$M_{12} + 2M_{23} = 0$

$M_{31} = 0$

$2M_{22} = 0$

$M_{13} = 0$

$a_{11} = 1 + \varepsilon m_{11}$

$a_{32} = 0 + \varepsilon m_{22}$

$a_{11} a_{32} = (1 + \varepsilon m_{11})(0 + \varepsilon m_{22})$

εm_{32} $= 0 + \varepsilon(0 m_{11} + 1 m_{22})$.

$0 + \varepsilon M_{21}$

$(1 + \varepsilon M_{22})^2 = 1 + 2\varepsilon M_{22}$

$$M = \begin{pmatrix} y & -2z & 0 \\ x & 0 & z \\ 0 & -2x & -y \end{pmatrix}$$

In SO_3

$M_{22} = 0$ from $\operatorname{tr} M = 0$

$\det A = 1$
$\downarrow$
$\operatorname{tr} M = 0$

M is 3-dimensional in any char K
$= \dim SO_3$

$\Rightarrow$ Our equations define a <u>reduced</u> group scheme with right root data, ... in any characteristic

N even or odd

It works over $\mathbb{Z}$:

Each Cartan datum $\longrightarrow$ A Hopf algebra $\mathcal{O}_{\mathbb{Z}}(G)$ over $\mathbb{Z}$ $\overset{\text{f.g.}}{\downarrow}$

$\mathbb{Z}[a_{ij}, \det(A)^{-1}]/I$ with compatible coproduct + antipode ...

such that it's torsion free (free) as an abelian group — flat over $\mathbb{Z}$.

- $K = \bar{K}$: $K \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Z}}(G) = \mathcal{O}_K(G)$ is the Hopf algebra of functions on the correct alg.

group G.

Ex. $G = Gl_n$: $\mathbb{Z}[a_{11}, \dots, a_{nn}, \det(A)^{-1}]$, $\Delta a_{ik} = \sum_j a_{ij} \otimes a_{jk}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

$R \leftarrow \mathcal{O}_{\mathbb{Z}}(Gl_n)$ $\quad\quad\quad\quad\quad\quad\quad \mathcal{O}_{\mathbb{Z}}(Gl_n)$

$\operatorname{Spec} R \rightarrow \operatorname{Spec} \mathcal{O}_{\mathbb{Z}}(Gl_n)$ $\leftsquigarrow$ "group scheme over $\operatorname{Spec} \mathbb{Z}$"

"R valued"

Functor $R \to$ groups $\quad\quad a_{ij} \mapsto r_{ij} \in R$

$R \to Gl_n(R)$ $\quad\quad \det(r_{ij}) \in R^\times$

$G = Sl_n$ $\quad \mathbb{Z}[a_{11}, \dots, a_{nn}, \det(A) = 1]$

$R \mapsto Sl_n(R)$

$G = \mathbb{G}_m = Gl_1$ $\quad R \to R^\times = \underline{\mathbb{G}_m}(R)$

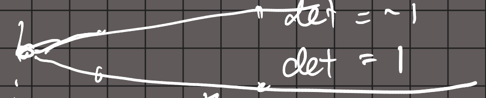
$G = \mathbb{G}_a$ $\quad R \to (R, +)$

Sp_{2n} : $\mathbb{Z}[a_{11}, \dots, a_{nn}] / I$ $\leftsquigarrow$ matrix entries of $A^T J_- A - J_-$

$R \to Sp_{2n}(R)$

O_N : $\mathbb{Z}[\underline{a}]/I \leftarrow$ eq. coefficients in $Q(Ax)=Q(x)$

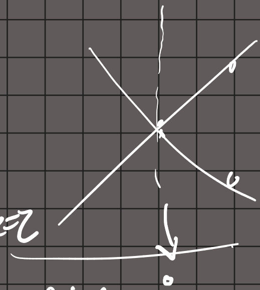
$R \to O_N(R)$



Spec $\mathbb{Z}$ (2) (3) ⋯ (7) ⋯⋯ (0)

SO_N

SO_N : add equation of a function that is 0 on SO_N and 1 on the other component if char ≠ 2



$x^2=y^2$

$x=0$

$y^2=0$

N odd : add $\det(A)=1$

N even : add something like $\frac{1+\det A}{2} = ad = 1$

$\frac{1-\det A}{2} = 0$

$R \to SO_N(R)$